Delocalization of Physical Laplacian Eigenvectors due to Degree-Volume Correlation

J. Yamamoto*, I. Bonamassa, and M. Pósfai,

Department of Network and Data Science, Central European University, 1100 Vienna, Austria.

Networks embedded in three-dimensional space under physical constraints occur naturally in various physical and biological systems, ranging from self-assembling systems and meta-materials to biological neural networks and vascular networks, to name a few. Motivated by the growing availability of three-dimensional maps of complex physical systems, recent studies [1–3] have unveiled unique structural and dynamical features that emerge due to the physicality of networks, such as topological entanglement [4, 5], bundling [6] and degree-volume correlation [3].

Recent research has shown that degree-volume correlations of the form $v \sim k^{\alpha}$ naturally arise in simple random-walk models of physical networks and influence their dynamics through a volume-weighted Laplacian [3]. This physical Laplacian, $\mathbf{Q}_{\mathrm{P}}$, has been derived for diffusive dynamics on physical networks in which inter-physical node coupling is much weaker than intra-physical node coupling [7], yielding the form $\mathbf{Q}_{\mathrm{P}} = \mathbf{V}^{-1/2}\mathbf{Q}_{\mathrm{G}}\mathbf{V}^{-1/2}$, where $\mathbf{V} = \mathrm{diag}(\{v_i\}_i)$ is a volume matrix whose diagonal entries are the volumes of physical nodes. Interestingly, many empirical networks show degree-volume correlations with $\alpha \sim 1$, suggesting certain natural selection towards linear correlations. Furthermore, degree-volume correlations in some empirical networks are noisy, with a variance of a few orders of magnitude.

To understand the consequences of linear degree-volume correlations and their broad noise distributions, we systematically investigate the roles of the degree-volume correlation strength, α, and noise, as well as of network topology in dynamics on physical networks by examining the spectral properties of physical Laplacians. We form a physical network by generating an abstract network and setting the volume of each node via $v \sim k^{\alpha}$. We then calculate the eigenvalues and eigenvectors of the physical Laplacian, $\mathbf{Q}_{\mathrm{P}}$, for a wide range of α values and various network topologies.

We begin by considering the simple case of a c-regular random (RR) graph with n nodes and a $(n+1)$-th node that attaches to d randomly selected nodes of the RR graph, where we assign unit volume to the nodes in the RR graph and volume $v \sim d^{\alpha}$ to the attached node. In this model, we calculate analytically the smallest non-zero and largest eigenvalues and their corresponding eigenvectors and show that the localization of eigenvectors in the attached node can be mitigated by its volume when α is close to one, resulting in delocalized eigenvectors. Next, we examine the physical Laplacian of more realistic network models with scale-free degree distributions. We observe the delocalization of the leading eigenvector for $\alpha \simeq 1$ and a re-entrant localization for $\alpha > 1$ due to degree heterogeneity. Finally, we study the eigenvectors of empirical physical networks under different levels of noise in degree-volume correlations and show that the eigenvectors undergo a transition from delocalized to localized states with increasing noise magnitude, both in continuous and discontinuous manner, depending on the underlying network topology and correlation strength.

From spreading processes to synchronization or percolation, hubs often play an outsized role in the dynamics on complex networks. Here we showed that in physical networks the emergent volume-degree correlation can suppress or mitigate the effect of the hubs. Although our motivation was to study physical networks, our results more generally apply to network-of-networks and multilayer networks whose layers have heterogeneous sizes.

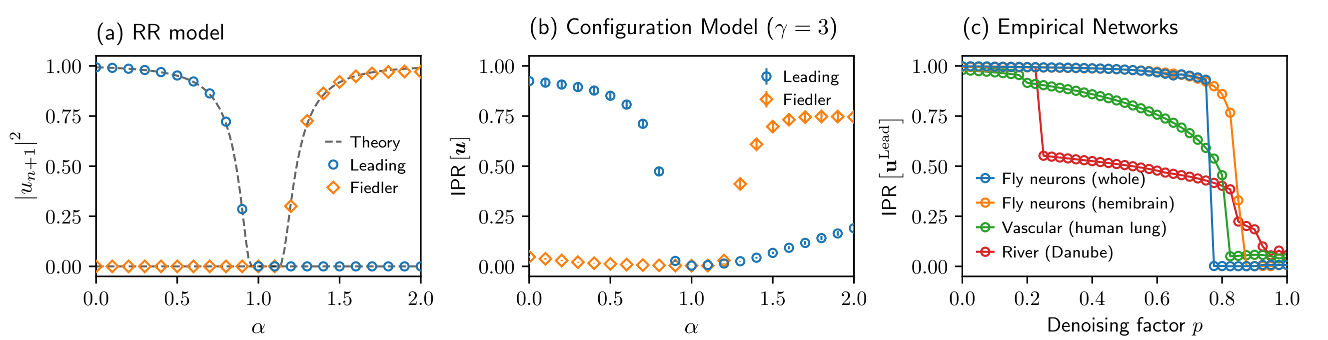

Figure 1: (a) The $(n+1)$-th component of Fiedler and leading eigenvectors of the RR model with $n = 10^5$, $c = 10$, $d = 316$. (b) The inverse participation ratios (IPRs) of the Fiedler and leading eigenvectors of configuration model ($n = 10^4$, $p_k \sim k^{-\gamma}$, $\gamma = 3$) with volumes assigned via $v \sim k^{\alpha}$. (c) The IPRs of the leading eigenvector of empirical physical networks with different denoising factor, p, where the volume of each node i is denoised as $\tilde{v}_i = v_i \exp\left[p \log\left\{v_{\mathrm{fit}}(k_i)/v_i\right\}\right]$.

[1] N. Dehmamy, et al. Nature **563**, 676–680 (2018).
[2] M. Pósfai, et al. Nat. Phys. **20**, 142–149 (2024).
[3] G. Pete et al., Nat. Comm. **15**, 4882, (2024).
[4] Y. Liu, et al. Nat. Phys. **17**, 216–222, (2021).
[5] C. Glover, et al. Phys. Rev. Lett. **133**, 077401 (2024).
[6] I. Bonamassa, et al. arXiv:2401.02579 (2023).
[7] S. Gómez, et al. Phys. Rev. Lett. **110**, 028701 (2013).

*jun.j.yamamoto@gmail.com