Multifractality of Scale-free Networks

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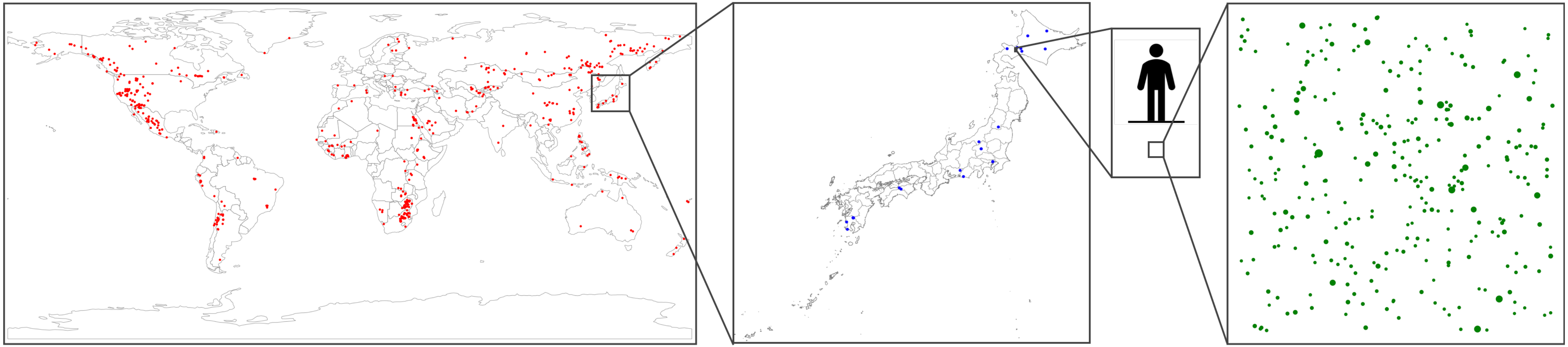
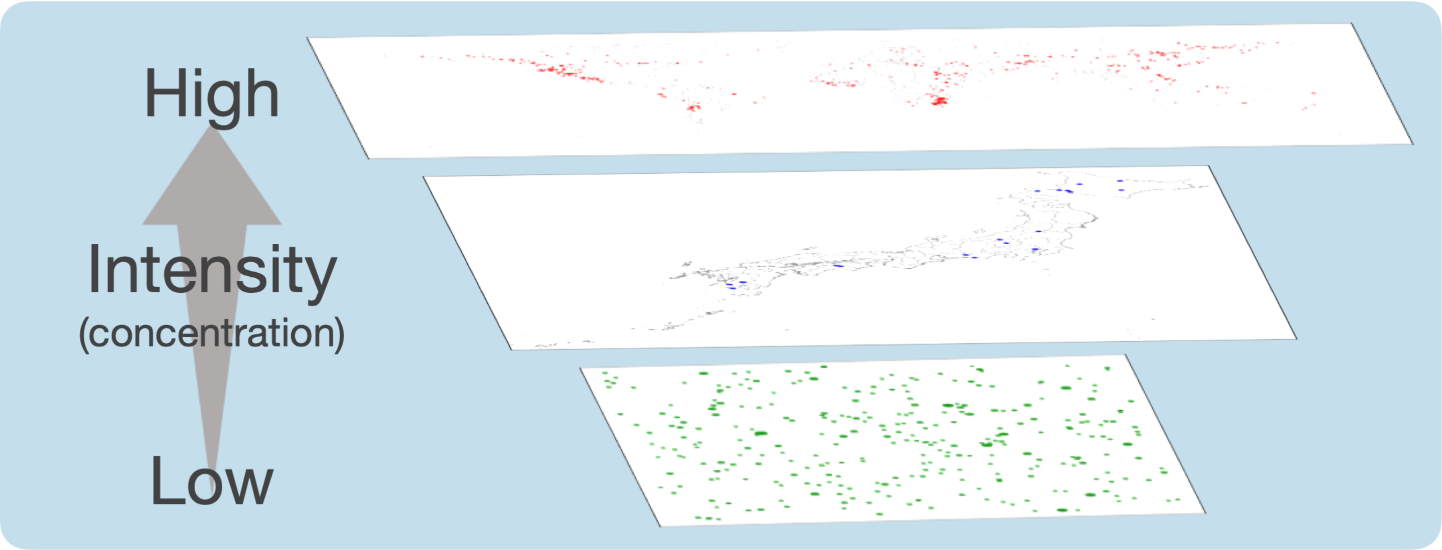
Multifractality



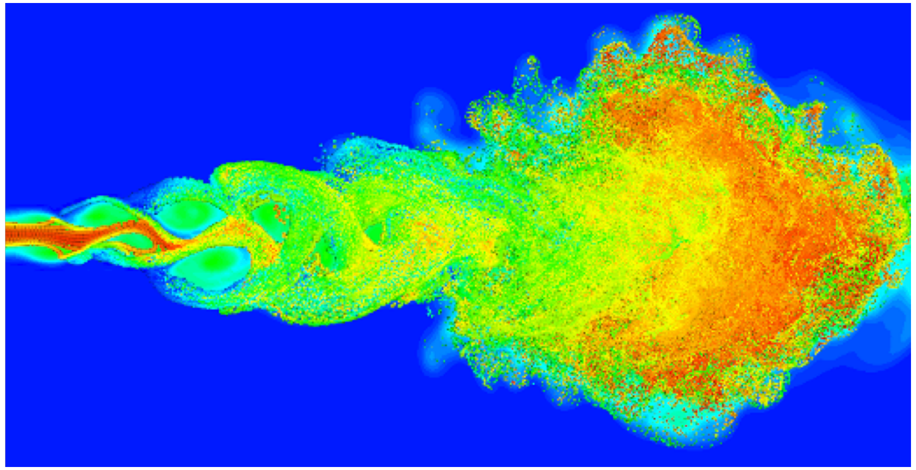


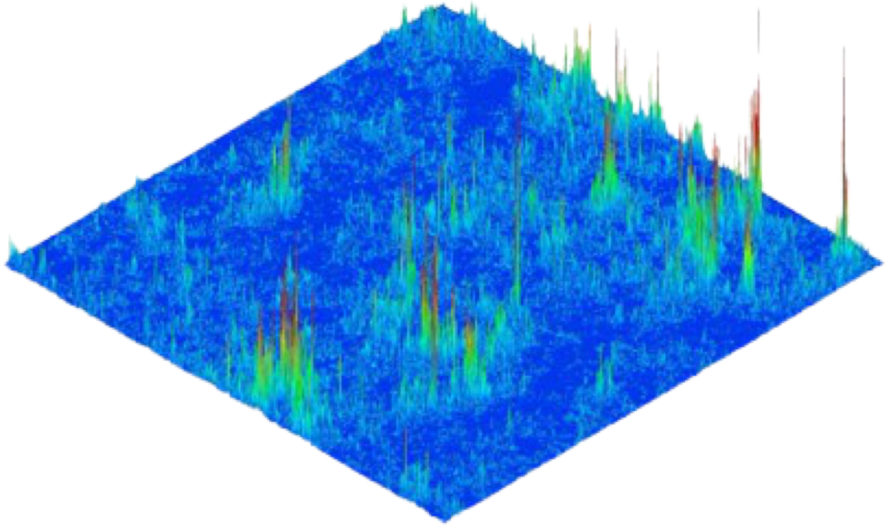
Multifractal systems:

The distribution of each intensity is characterised by its own fractal dimension.

Multifractality in Physics

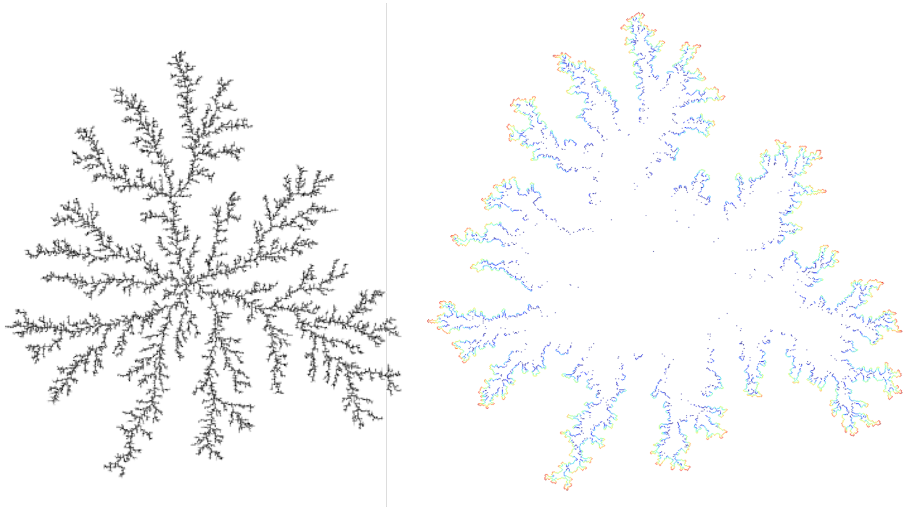
Energy Dissipation of Turbulence



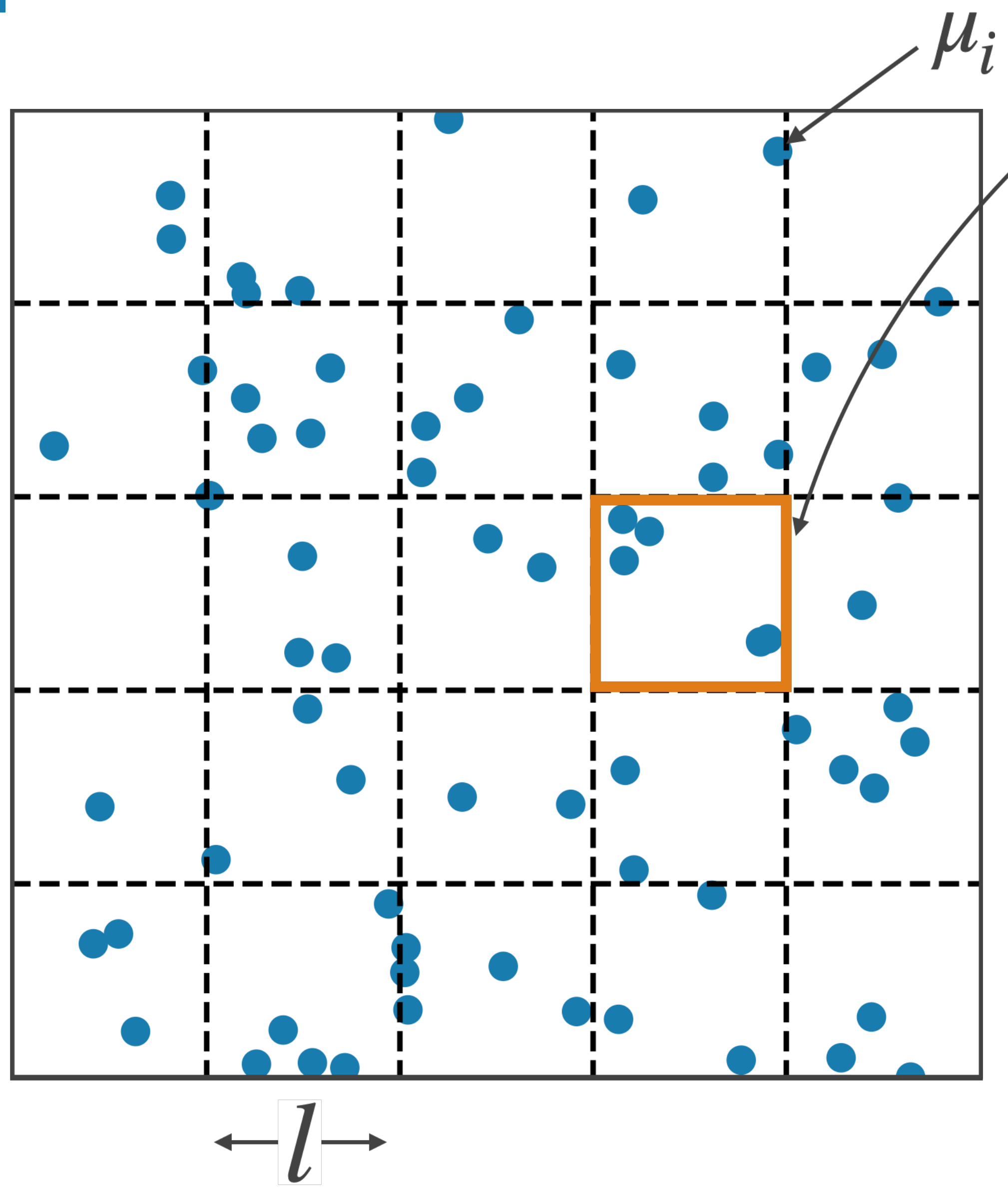


Critical Wavefunction in
Metal-Insulator Transition

Growth Probability in
Diffusion Limited Aggregation



Definition of Multifractality

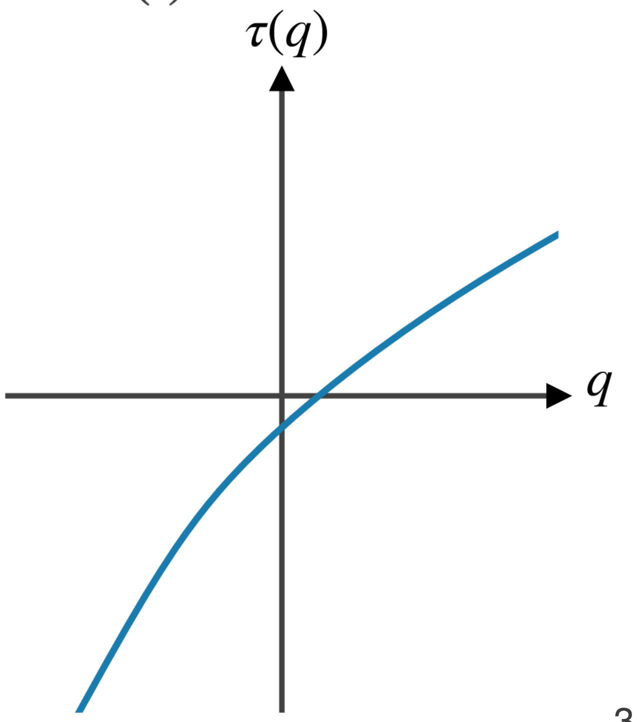


Box Measure $\mu_{b(l)} = \sum_{i \in b(l)} \mu_i$

q-th Moment $\langle \mu_l^q \rangle = \sum_{b(l)} \mu_{b(l)}^q \propto l^{\tau(q)}$

$\boxed{\tau(q) : \text{mass exponent}}$

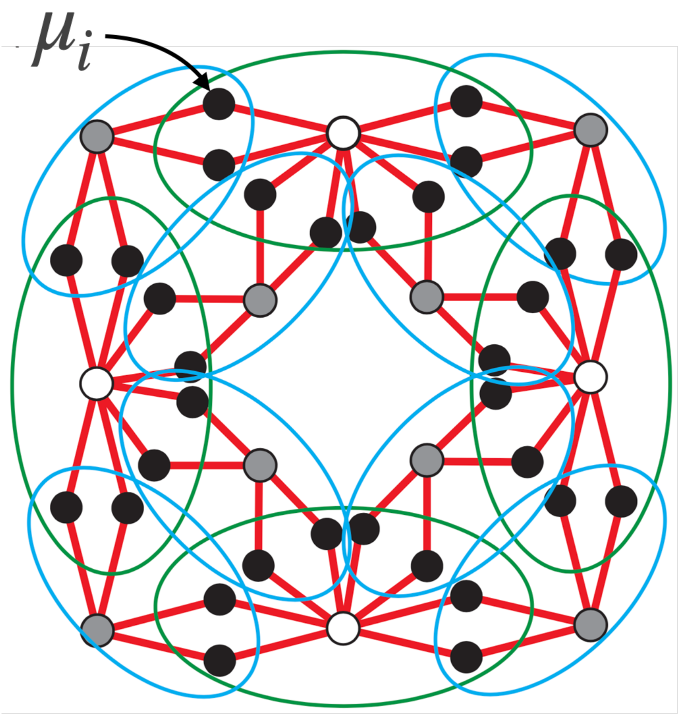
The distribution of μ_i is multifractal if $\tau(q)$ is a nonlinear function of q.



Multifractality of Complex Networks

[Furuya and Yakubo, Phys. Rev. E 84, 036118 (2011)]

Let μ be a measure on a network.

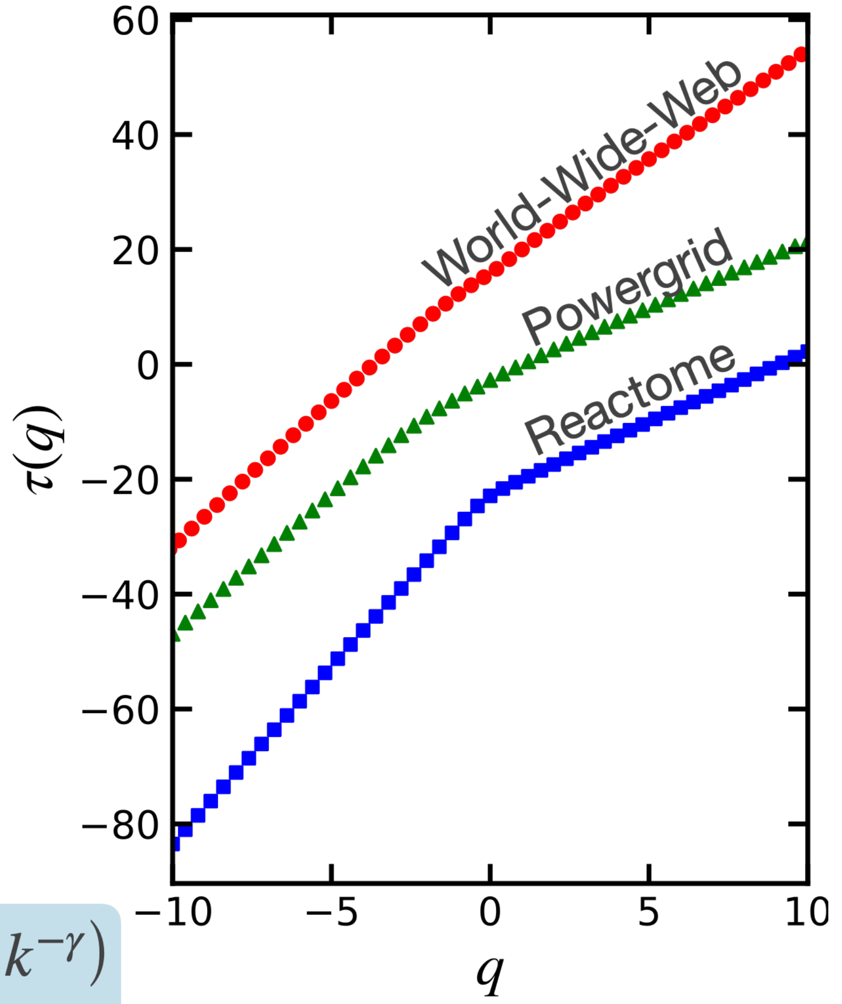


Box Measure $\mu_{b(l)} = \sum_{i \in b(l)} \mu_i$

q-th Moment $\langle \mu_l^q \rangle = \sum_{b(l)} \mu_{b(l)}^q$

The distribution of μ_i is multifractal if and only if

$$\begin{cases} \langle \mu_l^q \rangle \propto l^{\tau(q)}, \\ \tau(q) \text{ is a nonlinear function of } q. \end{cases}$$



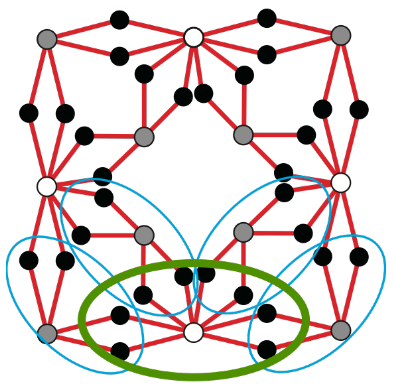
In case of fractal and scale-free networks, $\left(N_\mathrm{b} \propto l_\mathrm{b}^{-D_\mathrm{f}} \text{ \& } P(k) \propto k^{-\gamma}\right)$ even uniformly distributed measures μ_i exhibit multifractality!

Multifractality of Complex Networks

[Furuya and Yakubo, Phys. Rev. E 84, 036118 (2011)]

Multifractality of (u, v)-flower $(u, v:\text{even})$

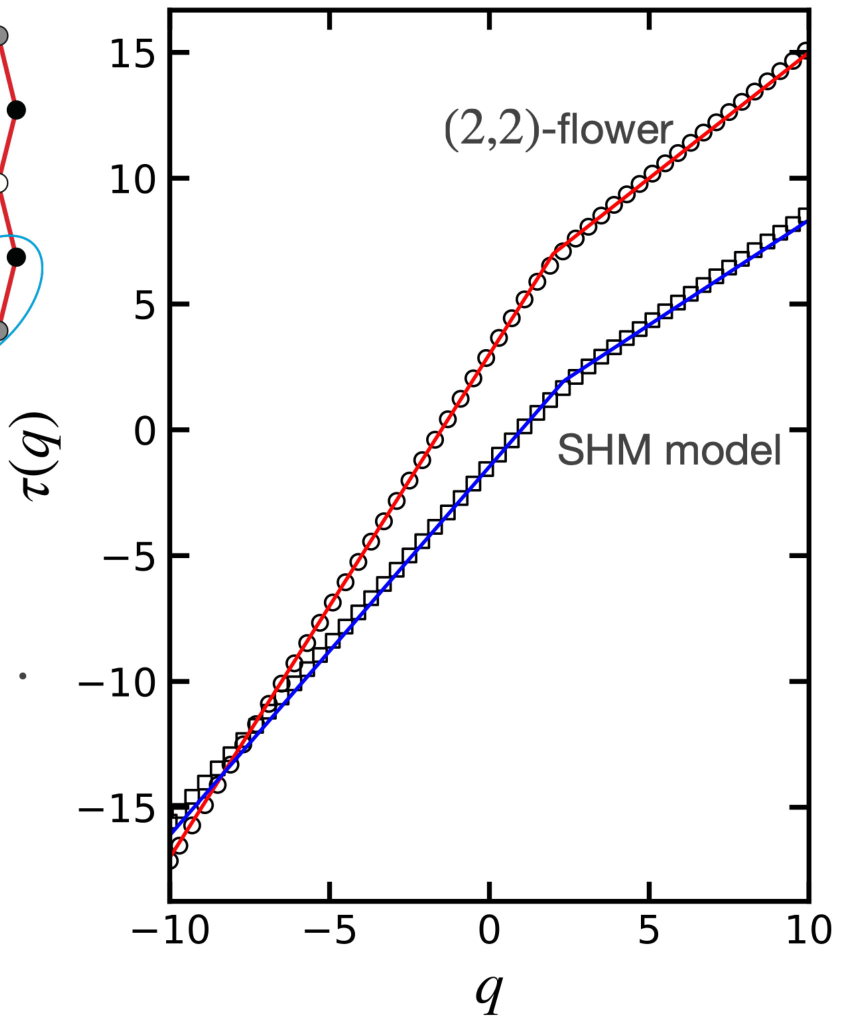
$$\tau(q) = \begin{cases} q\dfrac{\log(w/2)}{\log u} & \text{if} \quad q \geq \dfrac{\log w}{\log 2} \\ (q-1)\dfrac{\log w}{\log u} & \text{if} \quad q < \dfrac{\log w}{\log 2} \end{cases} \qquad (w = u + v)$$



$\nu_2(k_2) = 9$
$k_2 = 4$

More generally, FSFNs satisfying $\boxed{\nu_l(k_l) \propto k_l}$

possess the mass exponents $$\tau(q) = \begin{cases} qD_{\mathrm{f}}\dfrac{\gamma-1}{\gamma-2} & \text{if} \quad q \geq \gamma - 1 \\ (q-1)D_{\mathrm{f}} & \text{if} \quad q < \gamma - 1 \end{cases}.$$



Remaining Questions:

- Does a wider class of FSFNs also show bifractality?
- What is the relation between the bifractality and the structures of networks?

Goals

1. Examine whether a wider class of FSFNs exhibits bifractality.

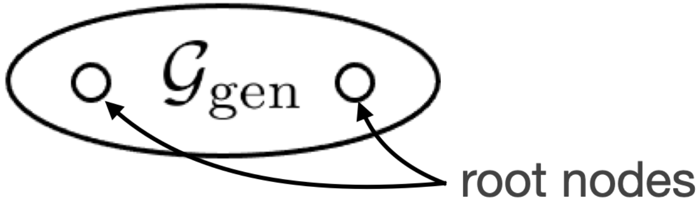
2. Identify the relation between the structures and the bifractality.

A General Model of Deterministic FSFNs

[Yakubo and Fujiki, arXiv:2109.00703.]

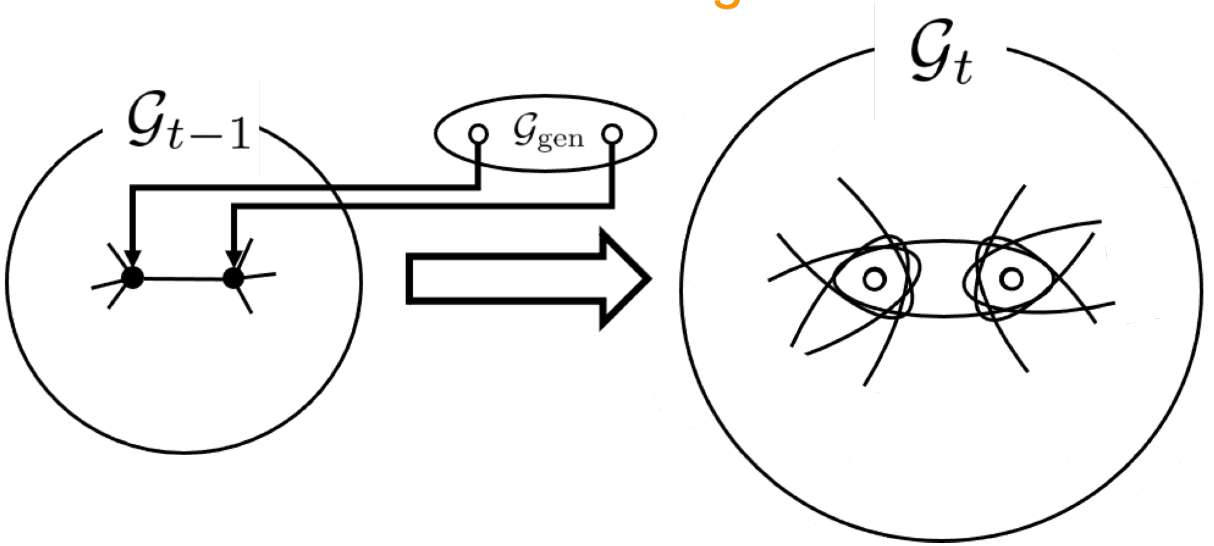
Preparation

$\mathcal{G}_{\mathrm{gen}}$: a generator



Operation

Replace all the edges of the previous generation in a manner that the root nodes correspond to the terminal nodes of the edges.



Conditions for $\mathcal{G}_{\mathrm{gen}}$

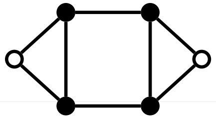
- $\mathcal{G}_{\mathrm{gen}}$ is connected.
- The root nodes are symmetric.
- The root nodes are not adjacent.
- The degrees of the root nodes is at least two.

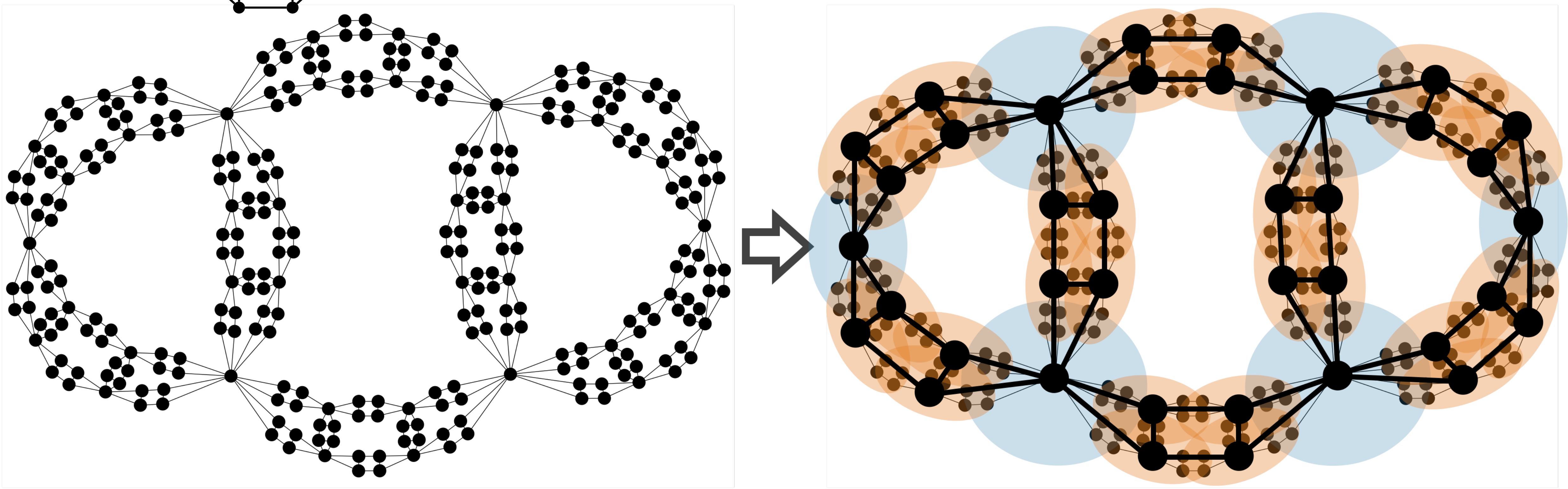
Scale-free Exponent $\gamma = \dfrac{\log m_{\mathrm{gen}}}{\log \kappa} + 1$

Fractal Dimension $D_{\mathrm{f}} = \dfrac{\log m_{\mathrm{gen}}}{\log \lambda}$

m_{gen}: # of edges in a generator
κ : degree of the root node
λ : distance between the root nodes

Multifractality of Deterministic FSFNs

Generator 



Renormalize a given network $\mathcal{G}_t$ in such a manner that the subgraphs compose the network $\mathcal{G}_{t'}$ of earlier generations. $(0 < t' < t)$

Any determinisitc FDFNs formed by this model satisfies the relation:

$$\boxed{\nu_l(k_l) = \frac{\langle \nu_l \rangle}{\langle k_l \rangle} k_l}$$

Mass Exponents of Deterministic FSFNs

Mass Exponent $\tau(q) = \begin{cases} qD_\mathrm{f}\dfrac{\gamma-2}{\gamma-1} & \text{if} \quad q \geq \gamma+1 \\ (q-1)D_\mathrm{f} & \text{if} \quad q < \gamma+1 \end{cases}$

$$\alpha = \frac{d\tau(q)}{dq}$$

Lipshitz-Hölder Exponent

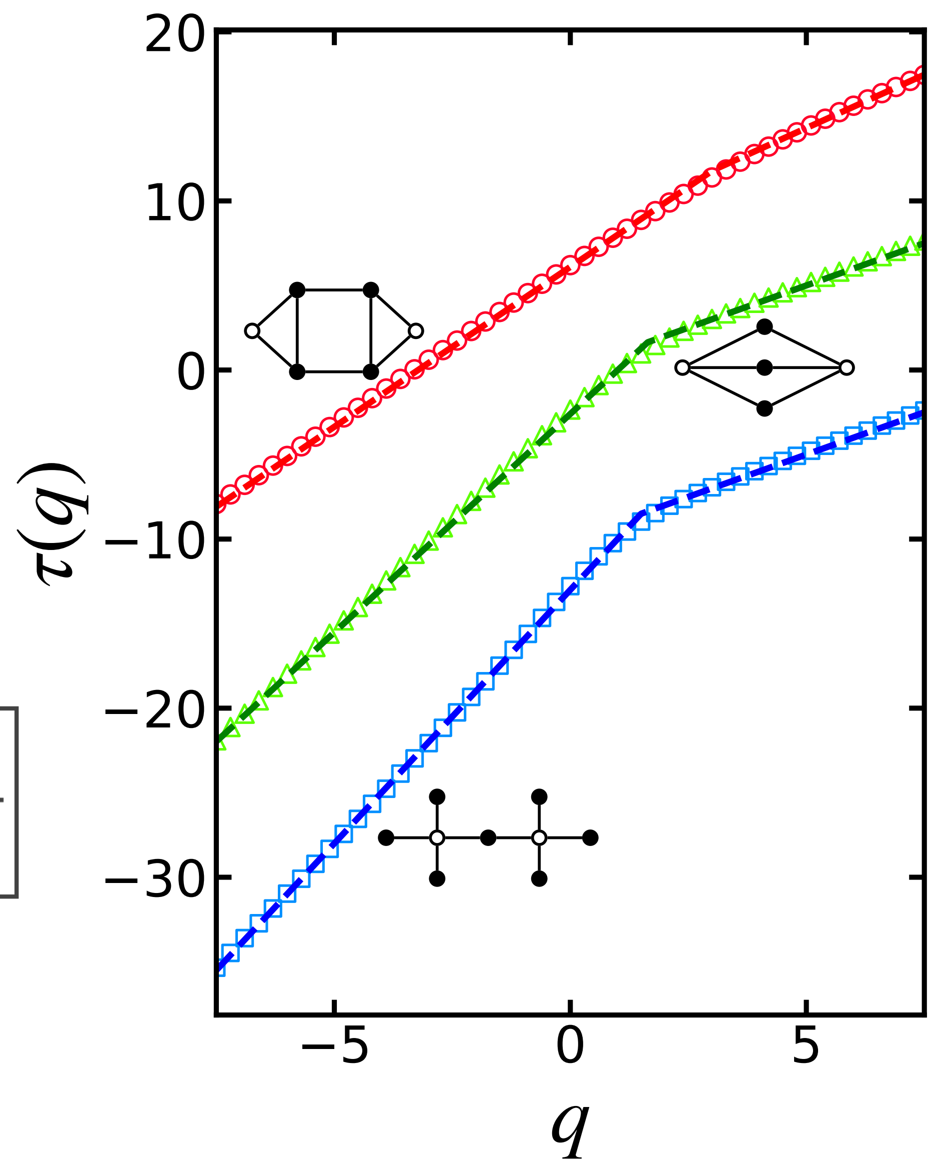
$$\alpha = \begin{cases} D_\mathrm{f}\dfrac{\gamma-2}{\gamma-1} & \text{if} \quad q \geq \gamma+1 \\ D_\mathrm{f} & \text{if} \quad q < \gamma+1 \end{cases}$$

Characterized by two distinct Lipshitz-Hölder exponents.

$$\boxed{\alpha = \lim_{l\to 0} \frac{\log M(l)}{\log l}}$$

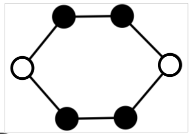
$$M(l) \propto l^\alpha$$

There exist two kinds of local structures within the network!

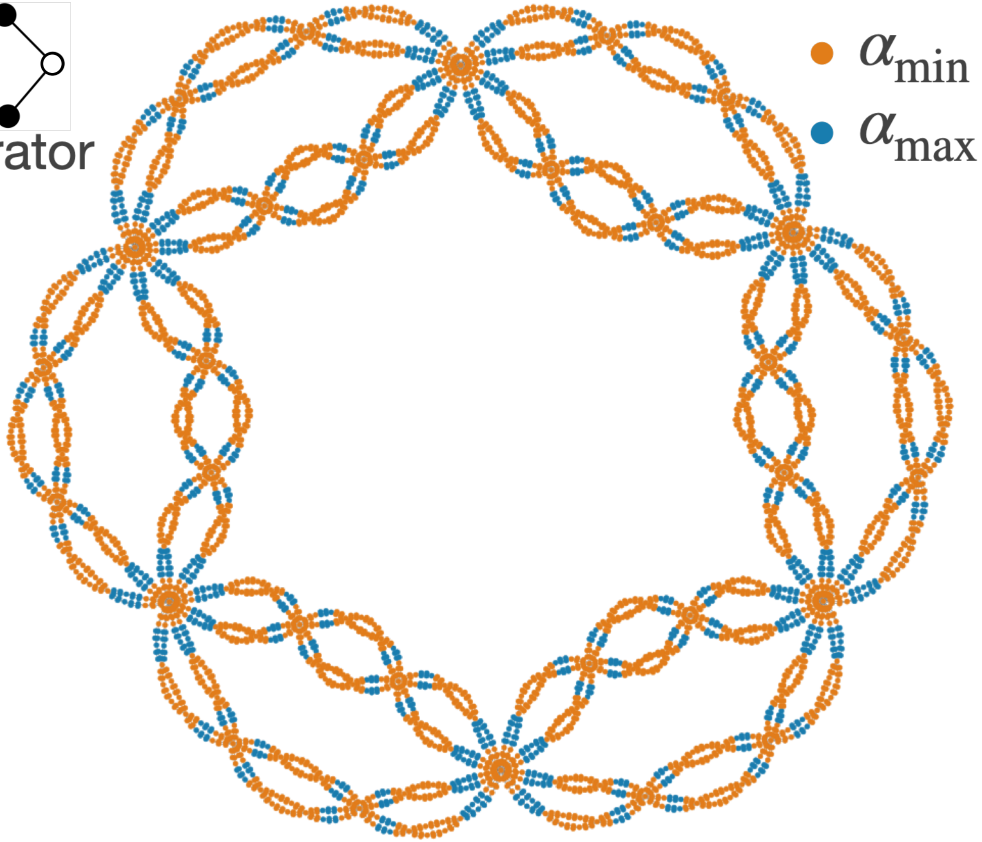


Lipshitz-Hölder Exponents of Deterministic FSFNs

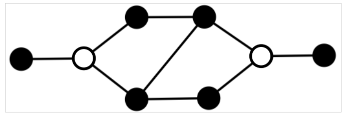
Lipshitz-Hölder Exponent $\boxed{\alpha = \lim_{l \to 0} \frac{\log M(l)}{\log l}}$



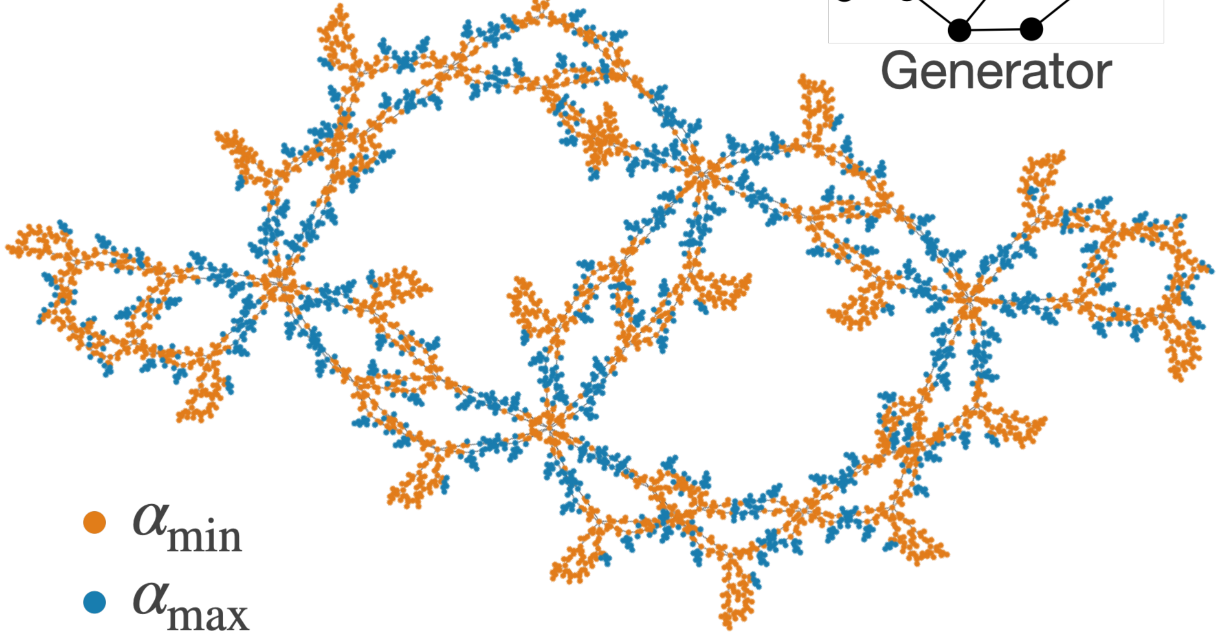
Generator



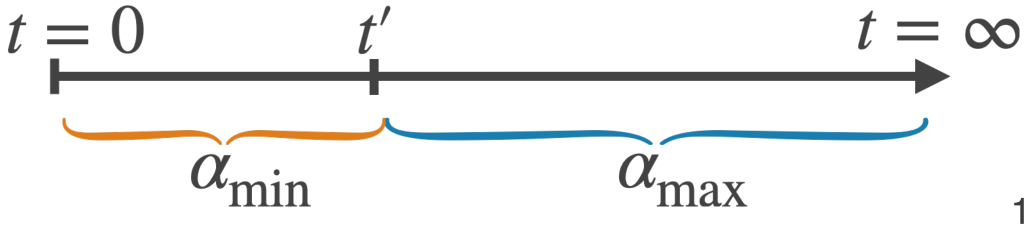
Classification of local structures using the K-means clustering algorithm



Generator



In thermodynamic limit $N \to \infty$,



Conclusion

1. Multifractal analysis of the generalized deterministic fractal scale-free network model is conducted.

2. We analytically and numerically show that the generalized deterministic fractal scale-free network model generates bifractal networks.

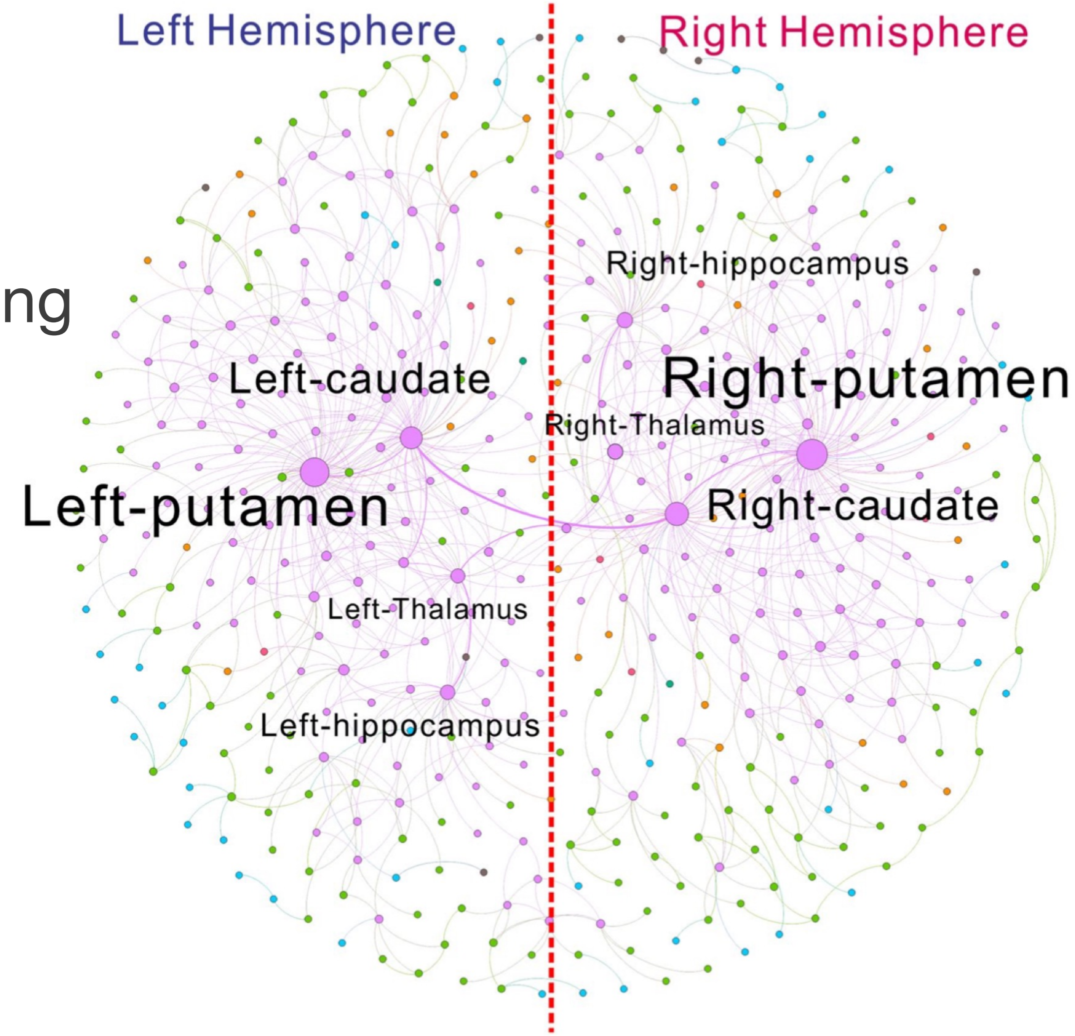
3. The two distinct Lipshitz-Hölder exponents correspond to local structures near the nodes which emerged in finite generation and those that are not.

Appendices

Application of Multifractal Analysis of Networks

[Y. Xue and P. Bogdan, Scientific Reports 7, 1 (2017)]

Identification of functioning parts in human brain



Lipshitz-Hölder Exponents of $(2,2)$-flower

Lipshitz-Hölder Exponent of $(2,2)$-flower

$$\alpha = \begin{cases} \dfrac{\log 2}{\log 2} = 1 & \text{if} \quad q \geq \dfrac{\log 4}{\log 2} = 2 \\ \dfrac{\log 4}{\log 2} = 2 & \text{if} \quad q < \dfrac{\log 4}{\log 2} = 2 \end{cases}$$