PSJ Spring Meeting 2022
(Online)

Bifractal Property of Stochastic Fractal Scale-Free Network Model

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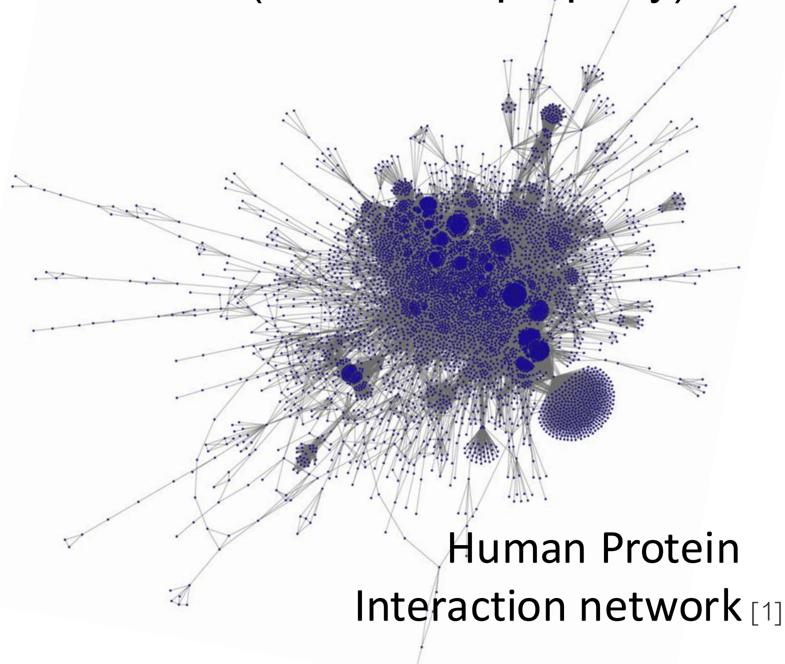
Complex Networks

Many real-world networks are large and hetergeneous.

- $N > 10^6$
- $P(k) \propto k^{-\gamma}$ for $k \gg 1$
  (Scale-free property)



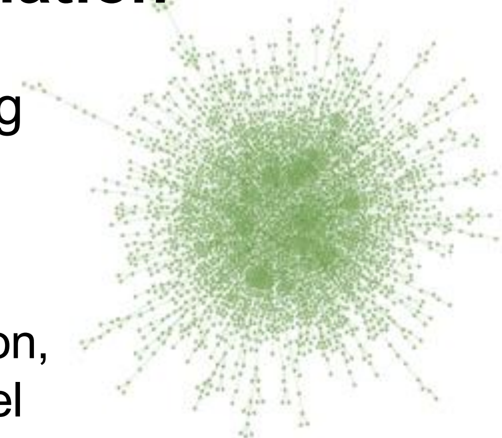
Human Protein Interaction network [1]

Approximation

No-correlation

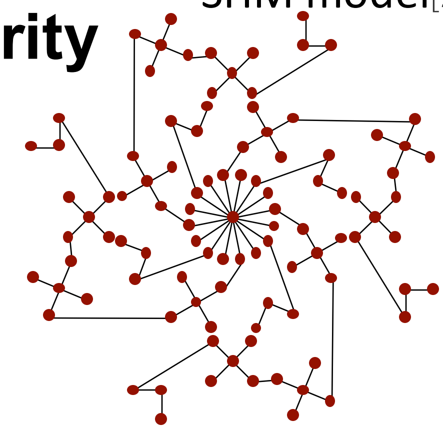
Generating Functions

e.g. :
percolation,
SIS model



Self-similarity

SHM model[2]



Fractal Scale-Free Networks (FSFNs)

$N_b \propto l^{-D_{\mathrm{f}}}$ (Fractal property)

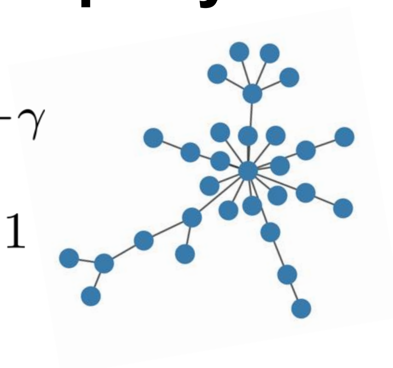
[1] G. Joshi-Tope et al., Nucleic Acids Research 33.suppl 1, D428-D432 (2005).
[2] C. Song, S. Havlin, and H. A. Makse, Nat. Phys. 2, 275 (2006).

Multifractality of Networks

Scale-Free Property

$$P(k) \propto k^{-\gamma}$$

for $k \gg 1$

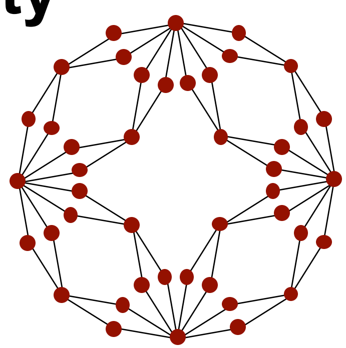


\+

Fractal Property

$$N_b(l) \propto l^{-D_{\mathrm{f}}}$$

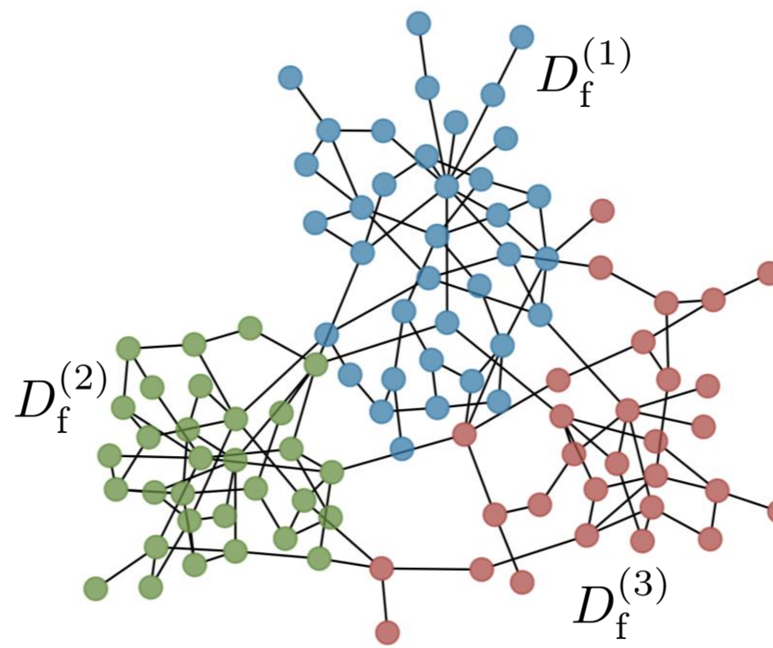
for $l \ll L$



(u, v)-flower [3]

Multifractal Property of Network Structures

Furuya and Yakubo,
PRE 84, 036118 (2011).

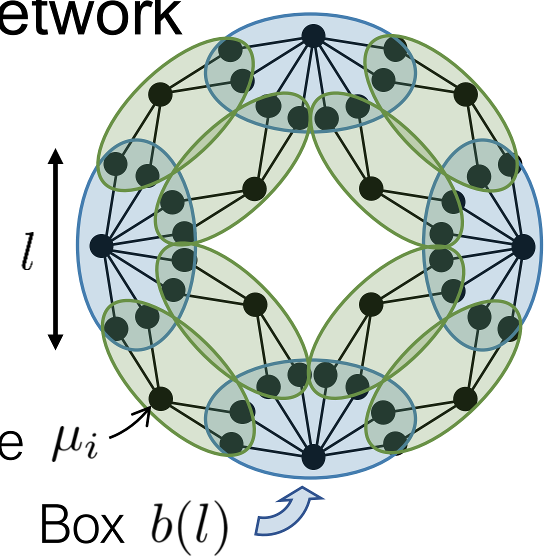


[3] H. D. Rozenfeld, S. Havlin, and D. ben-Avraham, New J. Phys. 9, 175 (2007).

Multifractality of Networks

Furuya and Yakubo, PRE 84, 036118 (2011).

1. Consider the box-covering of a network



2. Assign the node density measure to each node

$$\mu_i = \frac{1}{\sum_{b(l)} \sum_{i \in b(l)}}$$

3. Define the box measure

$$\mu_{b(l)} = \sum_{i \in b(l)} \mu_i$$

4. Compute q–th moment

$$Z_q(l) = \sum_{b(l)} \mu_{b(l)}^q$$

The network structure is **multifractal**

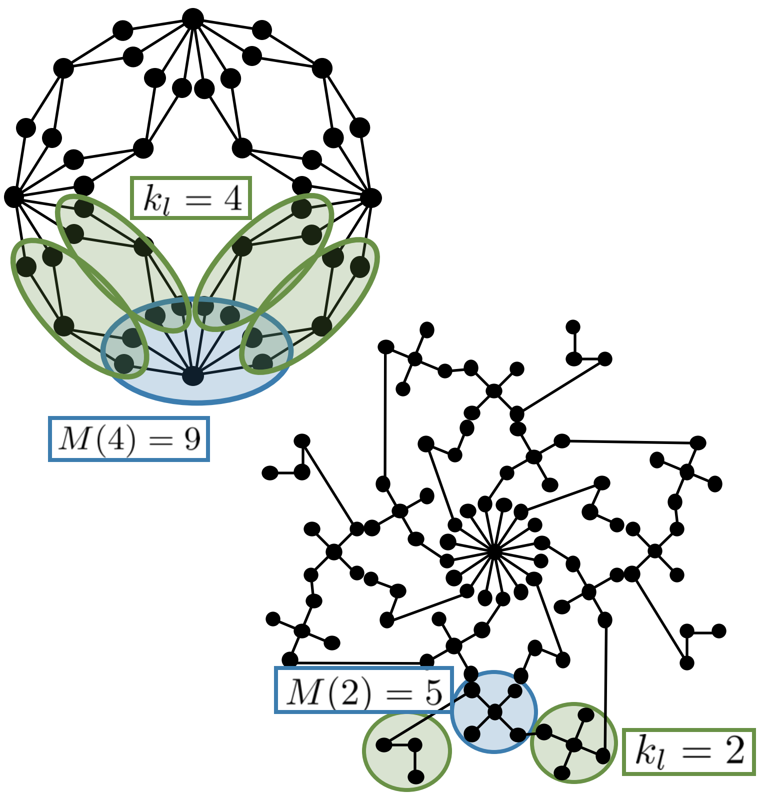
if $\begin{cases} Z_q(l) \propto l^{\tau(q)} \\ \tau(q) \text{ is nonlinear w.r.t. } q \end{cases}$

Bifractality of Some FSFNs

Furuya and Yakubo, PRE 84, 036118 (2011).

FSFNs satisfying

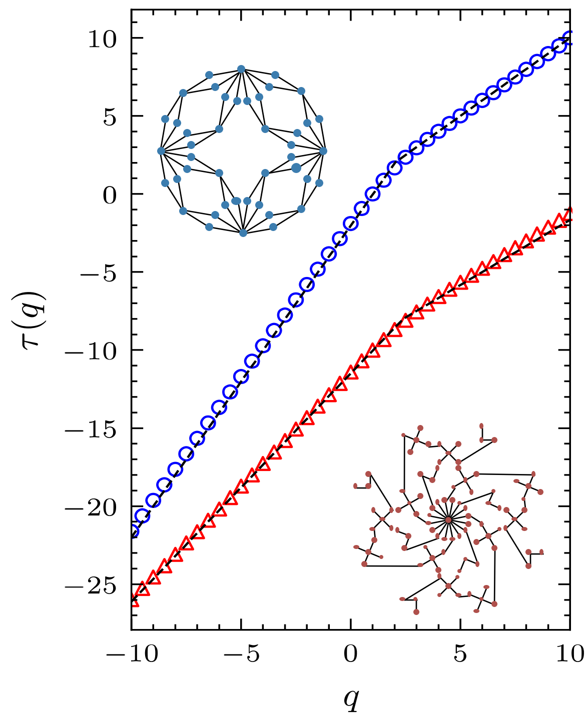
$$\boxed{M(k_l) \propto k_l}$$



Bifractality

$$q_0 \equiv \gamma - 1$$

$$\tau(q) = \begin{cases} qD_\mathrm{f}\dfrac{\gamma-2}{\gamma-1} & \text{if} \quad q \geq q_0 \\ (q-1)D_\mathrm{f} & \text{if} \quad q < q_0 \end{cases}$$



Unsolved Question

How general is the bifractal nature of FSFNs?

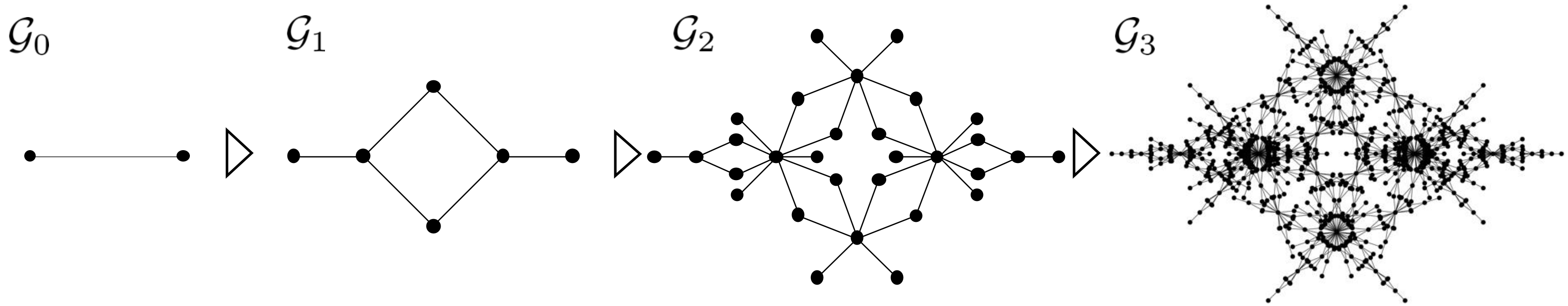
Goal

Investigate how general among FSFNs the bifractality is.

Bifractality of Deterministic FSFNs

A General Deterministic Model of Hierarchical FSFNs

Yakubo and Fujiki, PLOS ONE (to be published).

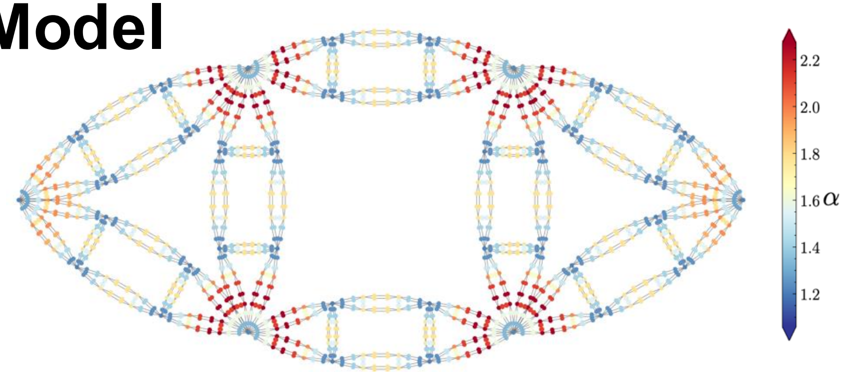


Bifractality of Deterministic Model of Hierarchical FSFNs

Yamamoto and Yakubo,
PSJ Autumn Meeting 2021.

$$q_0 \equiv \frac{\log m_{\mathrm{gen}}}{\log \kappa}$$

$$\tau(q) = \begin{cases} q\dfrac{\log m_{\mathrm{gen}} - \log \kappa}{\log \lambda} & \text{if} \quad q \geq q_0 \\ (q-1)\dfrac{\log m_{\mathrm{gen}}}{\log \lambda} & \text{if} \quad q < q_0 \end{cases},$$

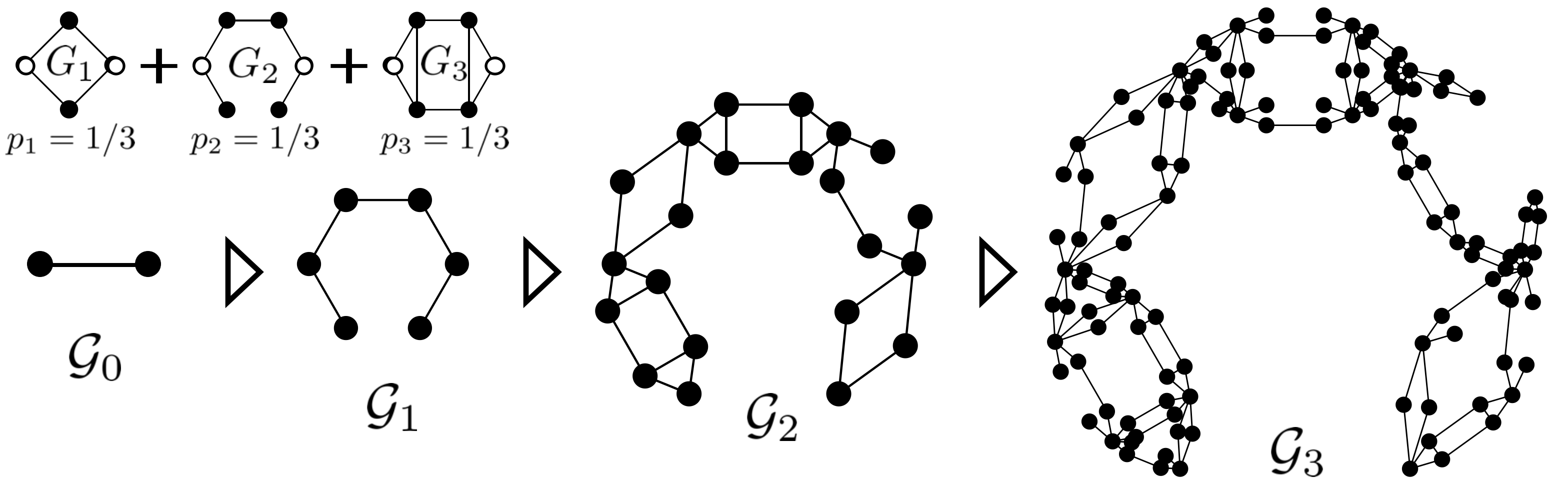


$$M_i(r) \propto r^{\alpha}, \quad \alpha = \begin{cases} \alpha_{\min} & \text{near hubs,} \\ \alpha_{\max} & \text{otherwise.} \end{cases}$$

Stochastic Model of HFSFNs

A General Stochastic Model of Hierarchical FSFNs

1. Prepare $\{G_i\}_{i=1}^n$ and $\{p_i\}_{i=1}^n$ where $\sum_{i=1}^n p_i = 1$.
2. Replace every edge of $\mathcal{G}_{t-1}$ by $G_j \in \{G_i\}_{i=1}^n$ with prob. p_j to form $\mathcal{G}_t$.



Scale-free Exponent $\gamma = 1 + \dfrac{\log \overline{m_{\mathrm{gen}}}}{\log \overline{\kappa}}$

Fractal Dimension $D_{\mathrm{f}} = \dfrac{\log \overline{m_{\mathrm{gen}}}}{\log \overline{\lambda}}$

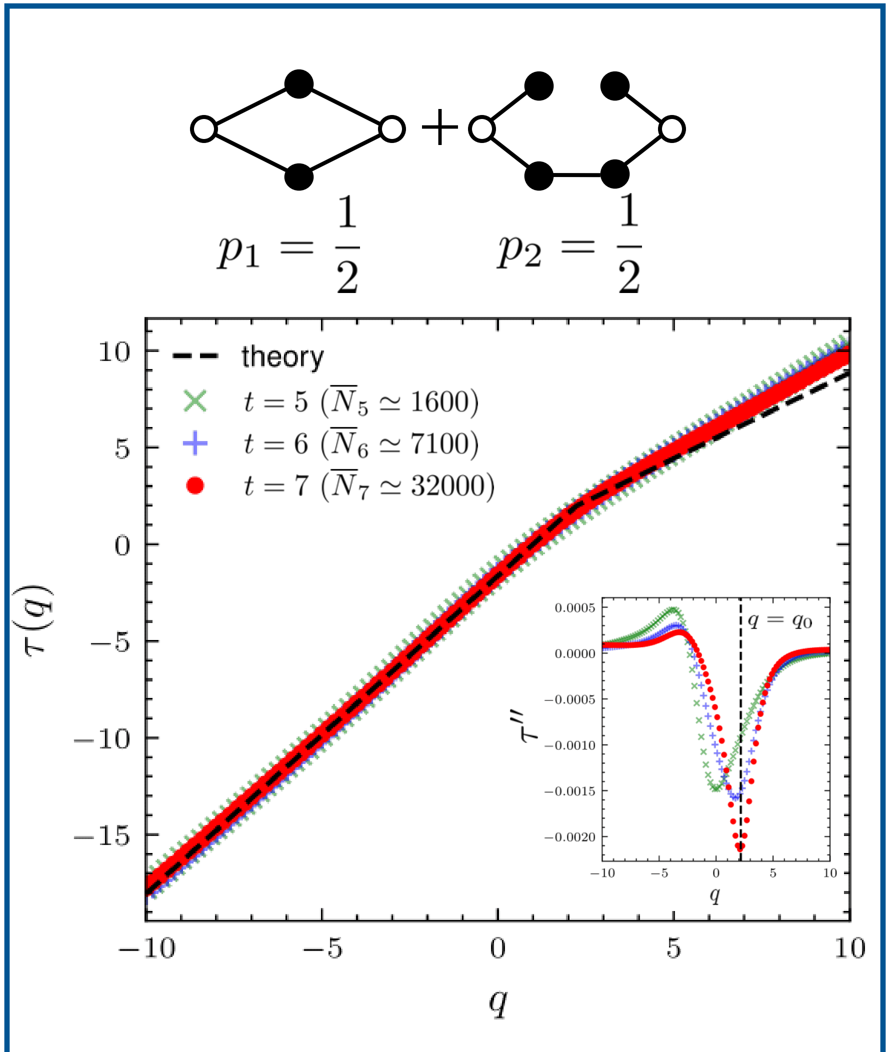
$$\overline{m_{\mathrm{gen}}} = \sum_i p_i m_{\mathrm{gen}}^{(i)}$$

$$\overline{\kappa} = \sum_i p_i \kappa^{(i)}$$

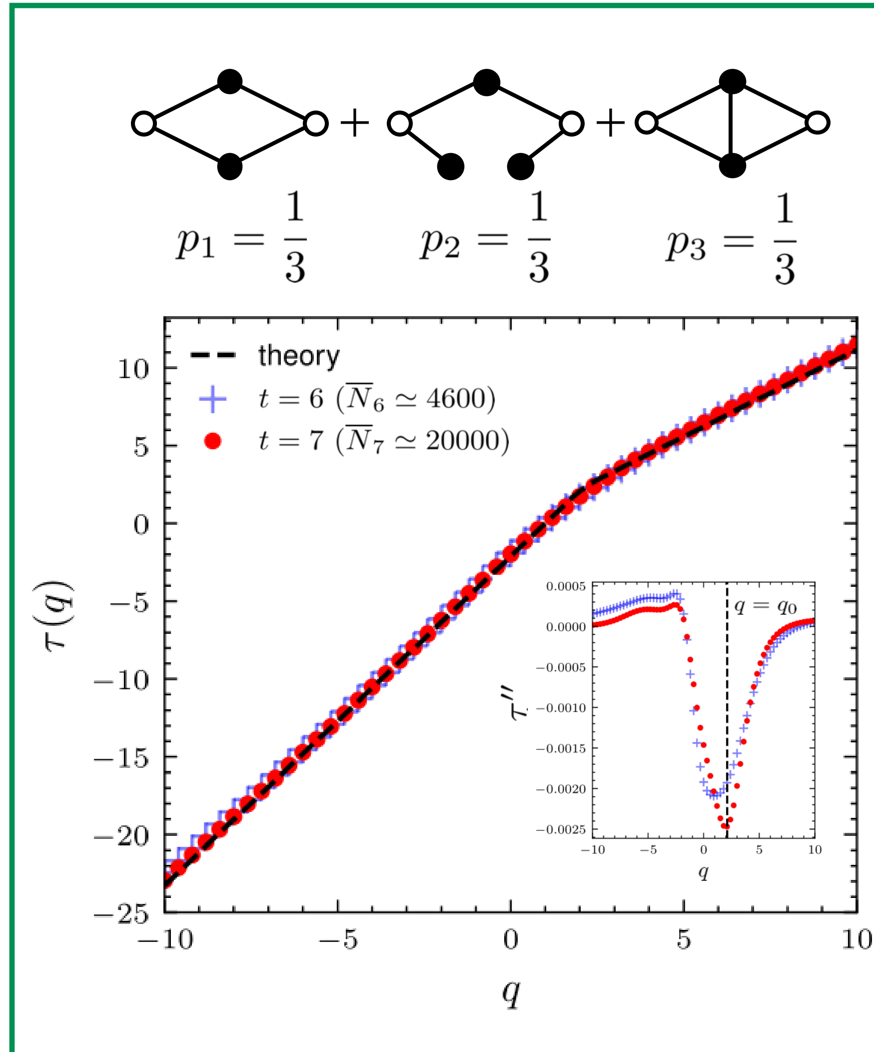
$$\overline{\lambda} = \sum_i p_i \lambda^{(i)}$$

Bifractality of Stochstic HFSFNs

$n = 2$

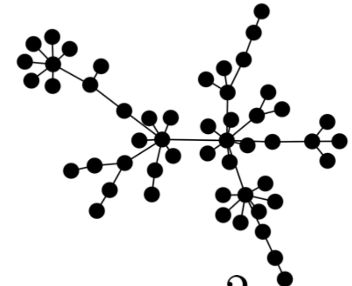


$n = 3$



Non-Hierarchical Model of FSFNs

Fractal Scale-Free Random Graphs (FSFRGs)

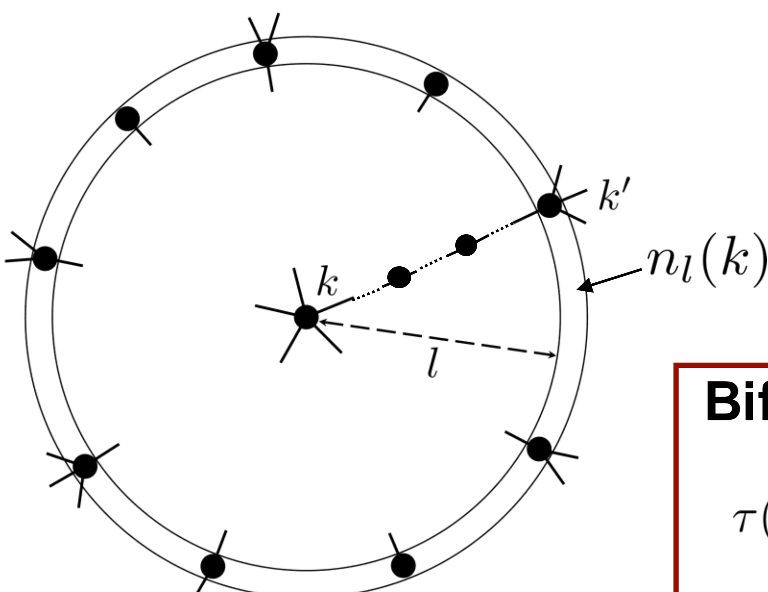


the giant component at the percolation transition point of scale-free random graphs

Scale-free Exponent $\gamma' = \gamma - 1,$ Fractal Dimension $D_{\mathrm{f}} = \dfrac{\gamma - 2}{\gamma - 3}$

P. Bialas and A. K. Ole, PRE 77, 036124 (2008).

Cohen et al. in *Handbook of Graphs and Networks* (Wiley-VCH, 2003).



$$M(k_l) = 1 + \sum_{l'=1}^{l} n_{l'}(k) = kl + 1$$

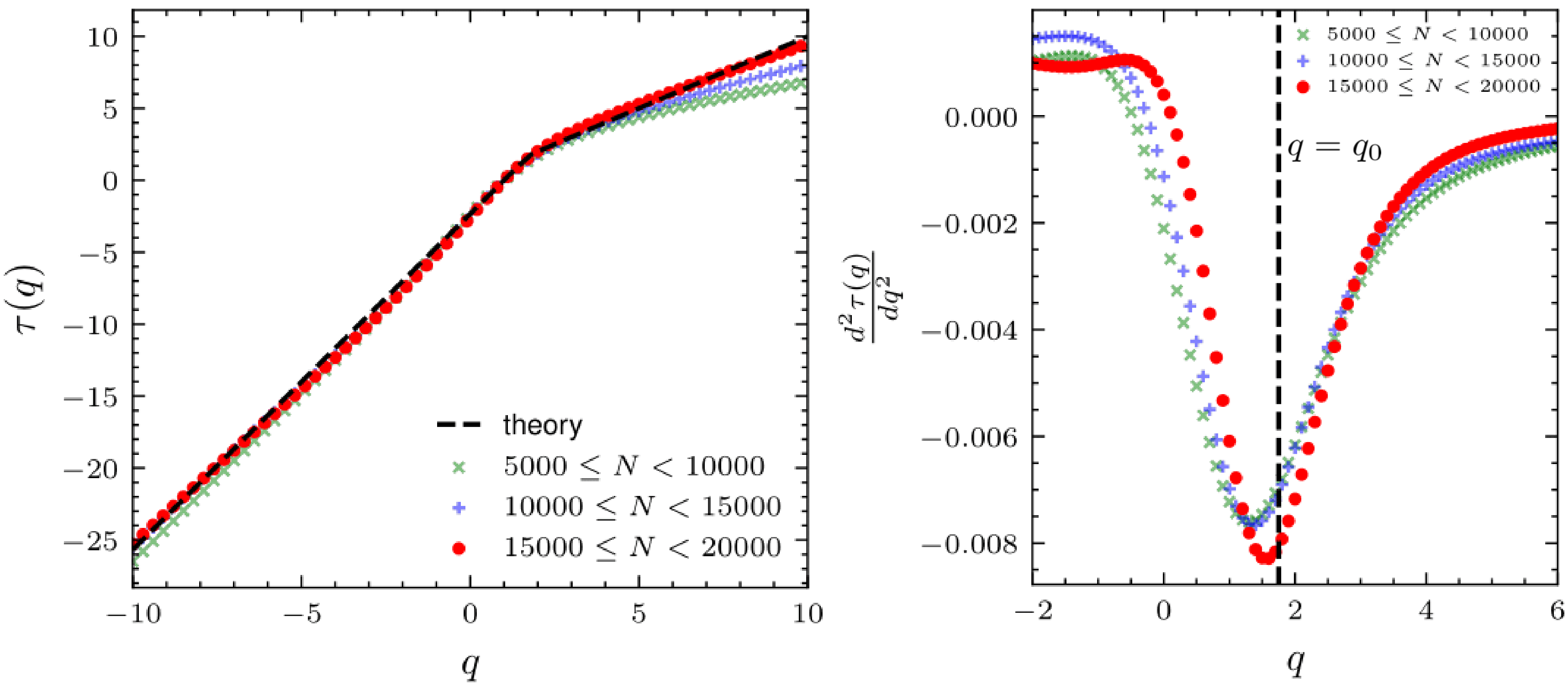
$$\boxed{M(k_l) \propto k_l}$$

Bifractality

$$\tau(q) = \begin{cases} q & \text{if} \quad q \geq q_0 \\ (q-1)\dfrac{\gamma-2}{\gamma-3} & \text{if} \quad q < q_0 \end{cases}, \quad q_0 \equiv \gamma - 2$$

Bifractality of FSFRGs

FSFRG with $\gamma' = 2.75,\ D_{\mathrm{f}} = \frac{7}{3}$



Summary and Discussion

Multifractal property of stochastic hierarchical and non-hierarchical FSFN models is studied.

- Generalized Deterministic Model of HFSFNs : **bifractal**
- Generalized Stochastic Model of HFSFNs : **bifractal**
- Non-hierarchical Fractal Scale-free Networks : **bifractal**

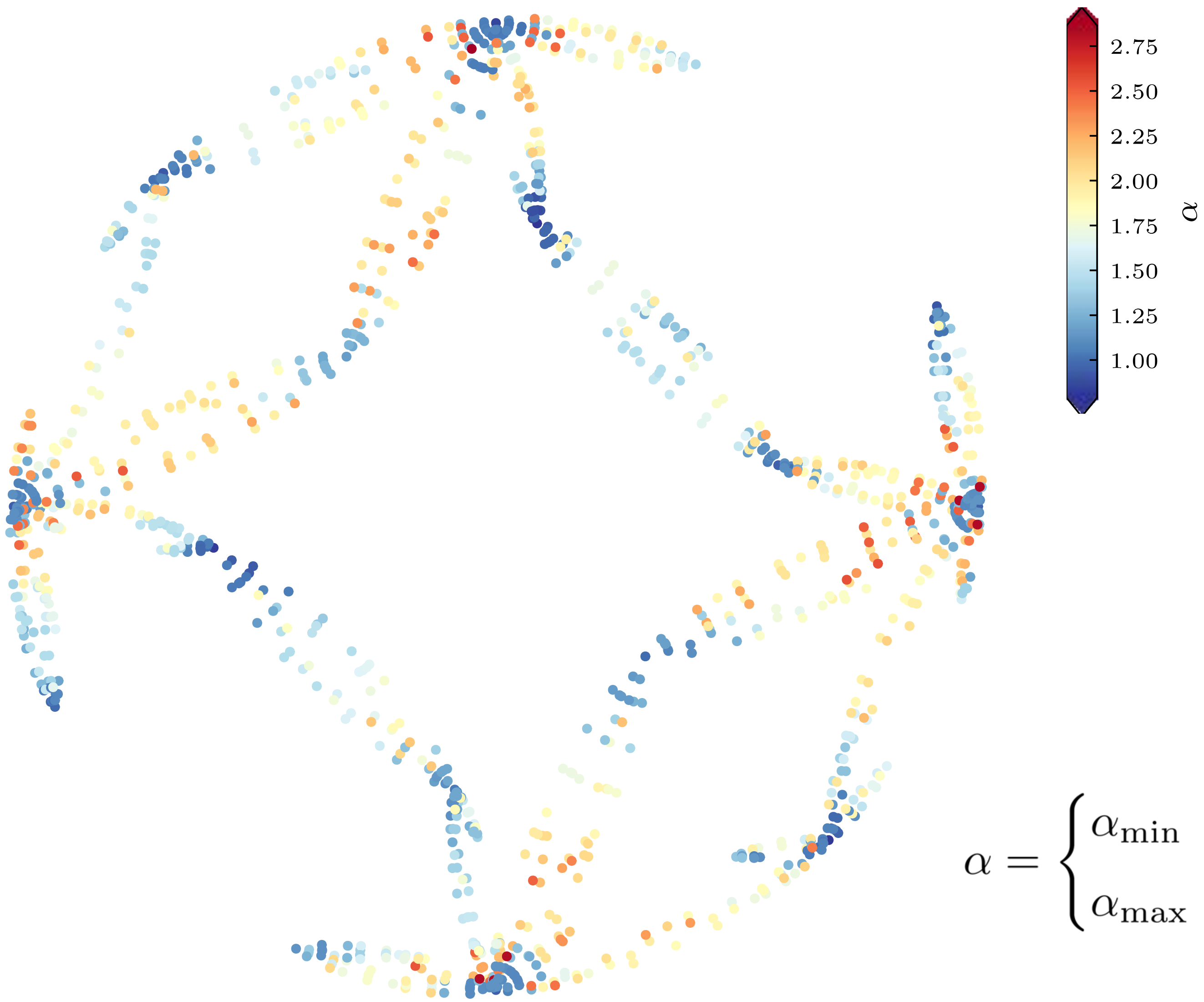
Conjecture: Any FSFN is **bifractal.**

Implications: two anomalous diffusions in FSFNs

e.g. information spreading, SIR model

Appendix

Correspondence with Structures

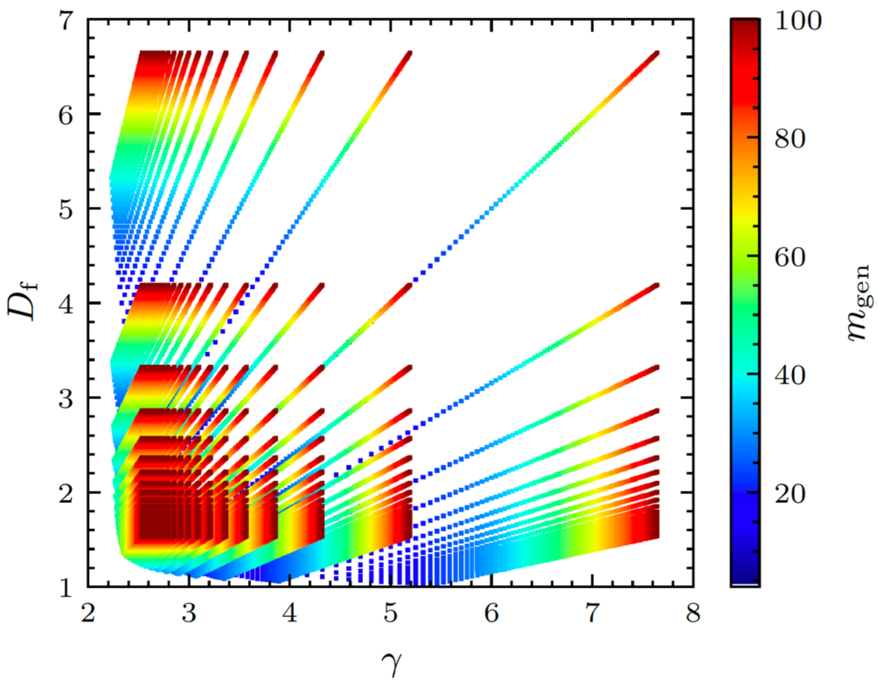


$$\alpha = \begin{cases} \alpha_{\min} & \text{near hubs,} \\ \alpha_{\max} & \text{otherwise.} \end{cases}$$

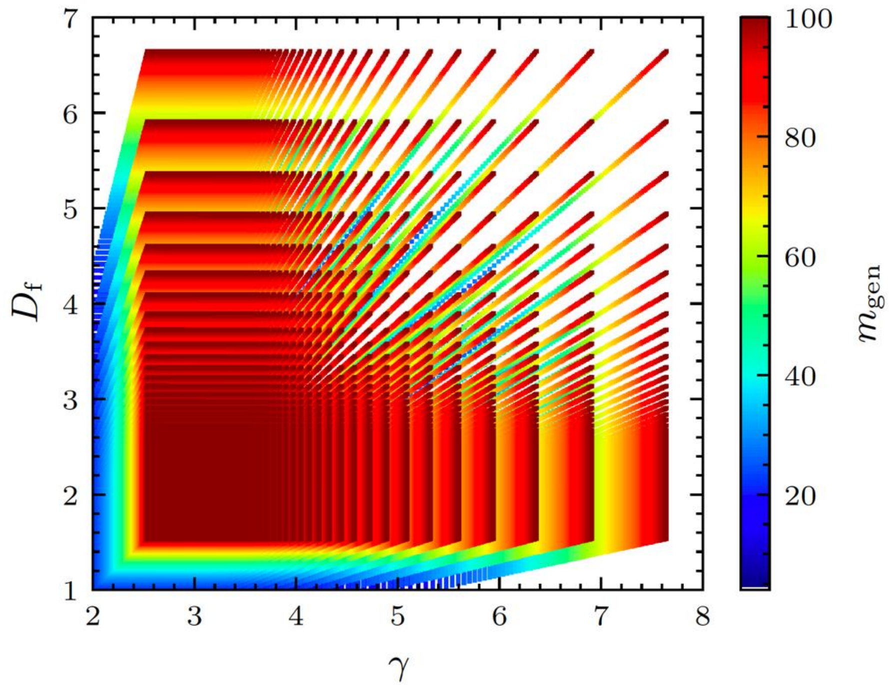
Phase Space of Stochastic Model

To see the stochastic model covers an extensive class of FSFNs, we visualize the phase space of the stochastic model.

Deterministic Model

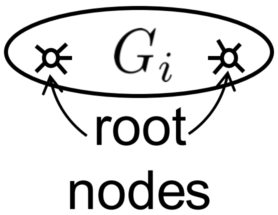


Stochastic Model



Stochastic Model of HFSFNs

Denote
- κ_i : degree of root nodes
- λ_i : distance between root nodes
- m_i^{gen}: number of edges



The degrees, the diameter, and the number of edges of $\mathcal{G}_t$ increase by the factors

$$\overline{\kappa} = \sum_i p_i \kappa^{(i)}, \ \overline{\lambda} = \sum_i p_i \lambda^{(i)}, \ \overline{m_{\mathrm{gen}}} = \sum_i p_i m_{\mathrm{gen}}^{(i)}$$

from those of $\mathcal{G}_{t-1}$, i.e.,

$$k(t) = \overline{\kappa} k(t-1), \ L(t) \sim \overline{\lambda} L(t-1), \ M(t) = \overline{m_{\mathrm{gen}}} M(t-1).$$

Thus,

Scale-free Exponent $\gamma = 1 + \dfrac{\log \overline{m_{\mathrm{gen}}}}{\log \overline{\kappa}}$

Fractal Dimension $D_{\mathrm{f}} = \dfrac{\log \overline{m_{\mathrm{gen}}}}{\log \overline{\lambda}}$