Bifractal property of scale-free networks

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A network is fractal with the fractal dimension D_{f} if the minimum number $N_{\mathrm{B}}(l)$ of subgraphs (boxes) with fixed diameter l required to cover the entire network is proportional to $l^{-D_{\mathrm{f}}}$. Previous studies revealed that many real-world networks are fractal, at least, on shorter scales than the average shortest-path distance [1,2]. However, if a network possesses the scale-free property in addition to fractality, a single fractal dimension may not be sufficient to fully describe the correlated structure of the network [3]. These networks exhibit a multifractal nature. Multifractal structures have been found in various fractal scale-free networks (FSFNs), and numerous efficient algorithms for analyzing the multifractal property of complex networks have emerged [4,5]. In particular, it has been clarified that an FSFN is always bifractal in which two local fractal dimensions suffice to characterize the fractal nature of the network, if we have the relation

$$\nu_b \propto k_b, \tag{1}$$

where ν_b is the number of nodes in a covering box b and k_b is the number of neighboring boxes of b [3]. The local fractal dimensions $d_{\mathrm{f}}^{\mathrm{min}}$ and $d_{\mathrm{f}}^{\mathrm{max}}$ are then given by

$$d_{\mathrm{f}}^{\mathrm{min}} = D_{\mathrm{f}}\left(\frac{\gamma-2}{\gamma-1}\right),$$
$$d_{\mathrm{f}}^{\mathrm{max}} = D_{\mathrm{f}},$$

where D_{f} is the global fractal dimension and γ is the degree exponent describing the scale-free property. Although several examples of bifractal networks have been presented so far [3,6], it remains unclear how common the bifractal property of FSFNs is.

In this work, we study the structural bifractality of extensive classes of FSFNs and conjecture that any FSFN possesses a bifractal structure characterized by two local fractal dimensions [7]. First, we show that hierarchical FSFNs formed by the single- and multi-generator models [8] exhibit the bifractal nature of their structures. Analytically predicted local fractal dimensions $d_{\mathrm{f}}^{\mathrm{min}}$ and $d_{\mathrm{f}}^{\mathrm{max}}$ are numerically confirmed. Through research on this class of FSFNs, it is elucidated that in the thermodynamic limit $d_{\mathrm{f}}^{\mathrm{min}}$ describes substructures around infinitely high-degree hub nodes and finite-degree nodes at finite distances from these hub nodes, while $d_{\mathrm{f}}^{\mathrm{max}}$ characterizes the local fractality around finite-degree nodes infinitely far from infinite-degree hub nodes. Even in a finite network, the local fractal dimension becomes close to $d_{\mathrm{f}}^{\mathrm{min}}$ in the vicinity of a hub node and close to $d_{\mathrm{f}}^{\mathrm{max}}$ around a non-hub node, as shown in Fig. 1. Next, we demonstrate the bifractality of non-hierarchical FSFNs by examining the giant connected component of a scale-free random graph at the percolation critical point and evaluating the conditional probability describing the long-range degree correlation. Finally, we demonstrate that the local fractality of some real-world FSFNs is also characterized by two local fractal dimensions $d_{\mathrm{f}}^{\mathrm{min}}$ and $d_{\mathrm{f}}^{\mathrm{min}}$. From the fact that a broad class of FSFNs exhibit bifractality, we conjecture that any FSFN is bifractal. Our findings are significant in providing a unified understanding of qualitative differences between various dynamics on FSFNs near hub and non-hub nodes.

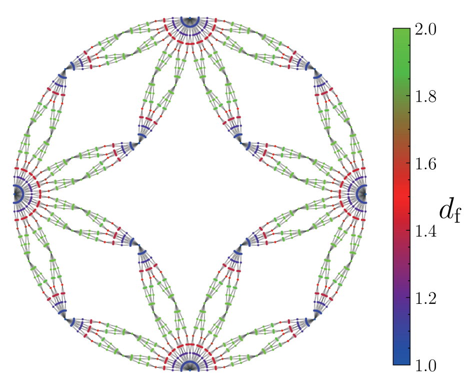

Figure 1: **Local fractal dimension.** Colors indicate values of local fractal dimension around each node.

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